On the Relation between Statistics of Scalar and Velocity Fluctuations in Developed Turbulence *

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For a developed turbulence it is shown how the statistics of a passive scalar, namely the probability density function of the temperature, can be related to the velocity intermittency. The results are compared with recent experimental findings on a grid-generated turbulence.

1. Introduction

It is well known [1] that the statistics of relevant quantities in a turbulent flow, such as the velocity or the density of active or passive scalars, can not be universal but depend on the specificities of the considered flow which can be comprised by the name "boundary conditions". Only the differences of such quantities between two points whose distance lies in the Kolmogorov [1] inertial or viscous subrange can pretend to some universality in the genesis of their probability density function (pdf). However, the apparent experimental ubiquity of some shapes for the pdf of passive [2–4] or active [5] scalars, in particular the Laplace or "exponential" shape led some authors [6, 7] to examine the existence of a "generic boundary condition" whose major influence would determine such shapes.

In this spirit, Shraiman and Siggia [7] recently discussed the pdf shape of passive scalar fluctuations in an externally imposed average gradient for this scalar. Though the characteristics they chose for the velocity field (single scale, "white noise") are not realistic for a turbulent flow, their detailed treatment of this model allows to stress the fruitful ideas which could be extended to more complicated realistic cases. This paper aims at going a step into this direction, trying to relate a scalar pdf to the velocity statistics in a developed turbulent flow.

Note that we have focused here on the particular problem of the statistics of the passive scalar fluctuations around its mean value. A far-reaching point is to deal with the relation between passive scalar fluctuations on different length scales and the corresponding velocity fluctuations. Detailed work on the latter point, showing how the structure function of passive scalars is linked to the structure function of the velocity field, is given in [8]. The evaluation of the shape of the pdf with the choosen length scale is discussed in [9] and [10].

2. Mixing Time and Conditional Probability

Shraiman and Siggia have pointed out that the distribution of the temperature of a fluid particle at $x = 0$ reflects the distribution of the positions this particle could have had a "mixing time" t_* before, via the average temperatures in those positions. Starting from this idea, we shall deduce the probability density function of temperature fluctuations in a developed turbulent flow submitted to a (passive) uniform average temperature gradient:

$$T(x) = T_0 + g x . \tag{1}$$

The instantaneous temperature is

$$T = T(x) + \theta . \tag{2}$$

As noted in [7], an estimate for the time a random strain field takes to stretch and fold a blob of size ξ to a scale on which molecular mixing predominates is

$$t_* \sim \gamma^{-1} \ln(\gamma \, \xi^2 / K) , \tag{3}$$

where γ is a typical Lyapunov exponent for velocity gradients and K is the temperature (scalar) diffusivity. In developed turbulence, γ must depend on the dissipated power per unit mass ε and the kinematic viscosity v, and thus must be close to $(\varepsilon/v)^{1/2}$ by simple

dimensional arguments. We shall take $\gamma^{-1} = A(v/\varepsilon)^{1/2}$, introducing the coefficient A.

On the other hand, fluid particles are carried with the large-scale typical velocity $u = (\langle u'^2 \rangle)^{1/2}$. The characteristic time for a noticeable change in u' along a Lagrangian path is of order l/u, where l is the integral scale. This time is much larger than γ^{-1}, and the particle can thus be considered as having a ballistic path $x = u' t_*$.

If the dissipated power ε were uniform in the flow, the temperature distribution would simply reflect the distribution of u', which is nearly gaussian [1],

$$P(u') \propto \frac{1}{u} \exp\left\{-\frac{u'^2}{2u^2}\right\}, \tag{4}$$

and thus

$$P(\theta | \varepsilon) \propto \frac{1}{u g t_*} \exp\left\{-\frac{\theta^2}{2 u^2 g^2 t_*^2}\right\}. \tag{5}$$

This result is not very far indeed from the corresponding intermediate result of [7]. At constant t_*, their result for $\langle \theta^2 \rangle$ is $g^2 V^2 \tau t_*$, where $V \cong u$ is the characteristic large scale velocity and τ the "white noise" cut off time, which cannot be very different from $(v/\varepsilon)^{1/2}$ in our context.

This comparison shows that $\xi \cong V\tau$ should be equivalent to the Taylor scale δ [1]:

$$\frac{u^2 v}{\varepsilon} = \frac{1}{15} \frac{u^2}{\langle (\partial u/\partial x)^2 \rangle} = \frac{\delta^2}{15}, \tag{6}$$

using the isotropic estimation of ε. As ξ enters in a logarithm, we shall take $\gamma \xi^2/K = R_\delta P_r$, where P_r is the Prandtl number and R_δ the Taylor scale based Reynolds number. Thus

$$t_* = A \left(\frac{v}{\varepsilon}\right)^{1/2} \ln(R_\delta P_r). \tag{7}$$

3. Probability Density Function of the Temperature Fluctuations

What preceeds is a plain transposition of the results of [7]. The difference comes now with the estimation of the distribution of "mixing times" t_*.

In developed turbulence, this distribution should come from the distribution of ε at the scale $x_* = u t_*$. The distribution of θ would then be

$$\Pi(\theta) = \int F(\ln \varepsilon) \frac{1}{\Theta(\varepsilon)} \exp\left\{-\frac{\theta^2}{2 \Theta^2(\varepsilon)}\right\} d \ln \varepsilon,$$

where

$$\Theta^2(\varepsilon) = A^2 u^2 g^2 \frac{v}{\varepsilon} \ln^2(R_\delta P_r). \tag{8}$$

This is to be compared with the distribution of velocity differences at a distance x, which can be written [11, 12]

$$\frac{1}{\sigma_x} P_x\left(\frac{\delta v}{\sigma_x}\right) = \int G_x(\ln \sigma) \frac{1}{\sigma} P_l\left(\frac{\delta v}{\sigma}\right) d \ln \sigma. \tag{9}$$

It has been shown that

– to a good approximation a gaussian ansatz can be used for G_x [13]:

$$G_x(\ln \sigma) = \frac{1}{\lambda_\sigma \sqrt{2\pi}} \exp\left\{-\frac{\ln^2 \sigma/\sigma_0}{2 \lambda_\sigma^2}\right\}, \tag{10}$$

– σ can be related to the average ε at the scale x by [11, 12, 14]:

$$\varepsilon \cong \frac{\sigma^3}{x}, \tag{11}$$

– in the inertial range of scales,

$$\lambda_\sigma^2 = \left\langle \ln^2 \frac{\sigma}{\sigma_0} \right\rangle \tag{12}$$

has a power law dependence on x [8, 9, 12]:

$$\lambda_\sigma^2 = \lambda_{0\sigma}^2 \left(\frac{x}{\eta}\right)^{-\beta}, \tag{13}$$

where $\eta = \left(\frac{v^3}{\varepsilon}\right)^{1/4}$ is the Kolmogorov length scale,

– the exponent β has a logarithmic dependence with the Taylor scale based Reynolds number R_δ at large R_δ [12, 15]:

$$\beta = \frac{\beta_0}{\ln(R_\delta/R_*)}. \tag{14}$$

The scale x_* at which ε is averaged is

$$x_* = \frac{1}{g} (\langle \theta^2 \rangle)^{1/2}. \tag{15}$$

Using (10) and (11), we see that

$$F(\ln \varepsilon) \cong \frac{1}{3 \lambda_\sigma(x_*) (2\pi)^{1/2}} \exp\left\{-\frac{\ln^2(\varepsilon/\varepsilon_0)}{18 \lambda_\sigma^2(x_*)}\right\} \tag{16}$$

which, using (8), gives the shape of the distribution of θ:

$$\Pi(\theta) = \frac{1}{2\pi\Lambda} \int \exp\left\{-\frac{\ln^2 \Theta/\Theta_0}{2\Lambda^2}\right\} \frac{1}{\Theta}$$
$$\cdot \exp\left\{-\frac{\theta^2}{2\Theta^2}\right\} d\ln\Theta \qquad (17)$$

with

$$\Lambda^2 = \frac{9}{4}\lambda_\sigma^2(x_*). \qquad (18)$$

For a moderate Reynolds number (18) and (15) allow to relate the temperature statistics to the velocity difference one. For very large Reynolds numbers the asymptotic form of (18) can be found at least within the hypothesis that $\beta\lambda_{0\sigma}^2$ tends toward zero when $R_\delta \to \infty$ [12]. Remarking that

$$\frac{\delta^4}{\eta^4} = 15\delta^2 \frac{u^2\nu}{\varepsilon} \frac{\varepsilon}{\nu^3} = 15 R_\delta^2 \qquad (19)$$

and using (8), (13), (15) and (18), we have

$$\langle \theta^2 \rangle = g^2 \frac{A^2}{15} \delta^2 \ln^2(R_\delta P_r) \qquad (20)$$

and

$$\Lambda^2 = \frac{9}{4}\lambda_{0\sigma}^2 e^{-(\beta_0/2)} e^{O(R_\delta)}, \qquad (21)$$

where $e^{O(R_\delta)}$ represents factors tending towards 1 when R_δ goes to infinity.

Note that (20) is not restricted to large R_δ. In this estimation we have neglected the fact that $\langle 1/\varepsilon \rangle \neq 1/\langle\varepsilon\rangle$. Our dimensional argument cannot distinguish between the two values. It will be refered to as the eventual intermittency correction and has a poor influence on Λ^2, (18).

4. Comparison with Experiments

In this section we summarize and compare to experimental results. We started from the Shraiman and Siggia [7] result that the shape of the temperature pdf is governed by the distribution of "mixing times" t_*. The new idea is to relate this distribution of t_* to the velocity intermittency.

There are no simultaneous measurements of the velocity intermittency and the temperature pdf in an average gradient. However, the grid flow used by Jayesh and Warhaft [4] in their study of temperature fluctuation has been investigated in [12] from the

point of view of velocity intermittency. We shall thus use these two experiments.

Let us first examine the mean squared fluctuations via the quantity $\langle \theta^2 \rangle / g^2 l^2$ as given in Fig. 3 of [4]. The overall agreement asks for $A^2 \cong 4$, which is reasonable. The agreement is not perfect for the variation versus d/M (d = distance from the grid, M = mesh of the grid) or versus the average velocity U. Our formula predicts a 30% increase of $\langle \theta^2 \rangle / g^2 l^2$ between $d/M = 36$ and $d/M = 102$. The real increase is close to a factor 2. We also predict a more pronounced decrease when U increases from 8.9 m/s to 15.7 m/s. However, a remark must be made here. Any theory based on universal properties of the flow will give a function of the single parameter R_δ for $\langle \theta^2 \rangle / g^2 l^2$. Thus the variations of this quantity versus U and d/M must be coherent through the corresponding variations of R_δ, which is not the case in this experiment.

Next we want to discuss the changing intermittent form of the pdfs of temperature. We have digitilized the data of Jayesh and Warhaft (Fig. 6 in [4]). Based on a χ^2-test [11–13] we looked for the best values of Λ^2 in (17). The result is shown in Figure 1. The changing
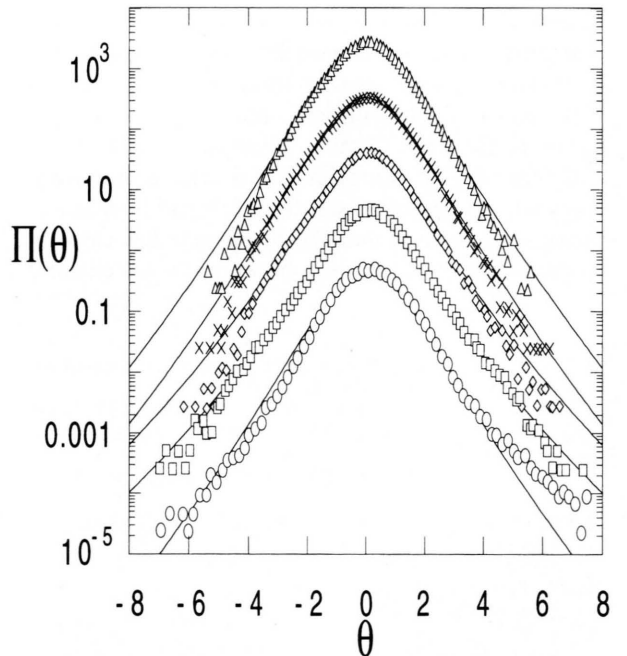

Fig. 1. Probability density function (pdf) of temperature at various d/M (after Fig. 6b of [4]). From bottom to the top d/M is 36.4, 62.4, 82.4, 102.4, and 132.4. The pdfs are normalized to its standard deviations. The solid lines show the best fits by (17) for appropriate Λ^2.
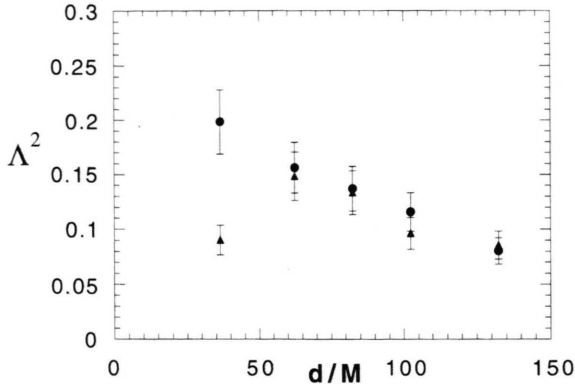
Fig. 2. Dependence of the intermittency parameter Λ^2 on d/M. The error bares are mainly due to our digitalization procedure. They were estimated from several subsequent repetitions of digitalization.

form with d/M is now parametrized by the Λ^2-dependence on d/M as shown in Figure 2. With the exception of one data point at $d/M = 36.4$, the tendency of a decreasing intermittency factor Λ^2 with increasing distance to the grid is found. (Just a short remark on this singular point: As can be seen in Fig. 1, for this experimental situation the fit of the pdf of the temperature is the worst. The pdf has a peculiar form, the maximum is quite round and the ailes are large. It may be that at this small distance from the grid and at this high velocity the turbulence is not perfectly homogeneous, in the sense of other measurements of Jayesh and Warhaft [4].) This finding of decreasing intermittency with increasing distance to the grid is in accordance with the fact that the turbulence decreases in this direction. A comparison between the velocity and

temperature intermittency can only be made at $d/M = 62$, $R_\delta = 54$. Using the results of [12] ($R_\delta = 69$), we see that the intermittency factor Λ^2 is 0.15. From our approach we expect it to correspond to a velocity intermittency factor λ_σ^2 of 0.07 at a scale $r \cong 28\eta = (\langle\theta^2\rangle)^{1/2}/g$. With similar conditions, [12] gives $\lambda_\sigma^2 \cong 0.04$ at this scale. We consider this as a partial agreement, taking into account that the two measurements were not performed simultaneously. We insist on the importance of characterizing simultaneously the velocity and temperature intermittencies.

5. Conclusion

We have presented an explanation how intermittent pdfs of a passive scalar may be related to intermittent pdfs of the velocity field. Overall we find quite satisfactory accordances. Definitely our results can only be taken as indicative, because they were verified only with quite low Reynolds number turbulence. Further conclusions on the validity of our approach desire further experimental work. High Reynolds numbers and simultaneous measurements of the velocity and a passive scalar would be highly interesting.

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